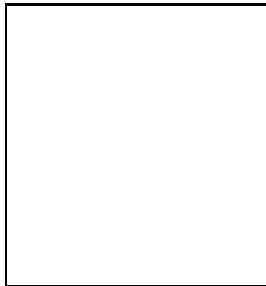# UNDERSTANDING THE 1-SKYRMION

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It is shown how the appearance of the 1-skyrmion on the three-sphere of radius L can be understood quantitatively by analyzing spectrum of the Hessian at the identity solution. Previously, this analysis was done qualitatively by N. Manton [7] which used a conformal deformation of the identity solution. Analysis of critical appearance of solutions in the Skyrme model on the three-sphere may be useful to understand similar phenomena in more complicated equations of mathematical physics.

1 Some General Remarks

In this paper static solutions of the $SU(2)$ Skyrme model on the three-sphere of radius R as a physical space are discussed. In a sense, this model includes the Skyrme model on the flat space $\mathbb{R}^3$ in the limit $R \to \infty$.

Historically, the Skyrme model was introduced to physics as a mesonic fluid model describing nuclear matter, and was proposed by T.H.R Skyrme in 1954 [1]. In agreement with the Yukawa picture, nuclear interactions are mediated by π-meson fields that are regarded as constituents of nucleons. The idea of Skyrme was to imagine such fields as an incompressible mesonic fluid. The state of such a fluid is defined by a scalar density and a direction in the isospace. Nucleons are assumed to be immersed into a mesonic liquid droplet, freely moving inside it forming a nucleus. In this effective meson field theory, low-energy baryons emerge as solitonic fields. The basic chiral field in that model is the $SU(2)$-valued scalar field U, which is related to the singlet meson field σ and the triplet pion field $\vec{\pi}$ by

$$U(\vec{x}) = \mathbf{f}_\pi^{-1}(\sigma(\vec{x}) + i\vec{\sigma} \circ \vec{\pi}(\vec{x})), \qquad (\sigma)^2 + (\vec{\pi})^2 = \mathbf{f}_\pi^2.$$

The model is characterized by two constants: $\mathbf{f}_\pi$, the pion decay constant and $\mathbf{e}$, determining the strength of the fourth-order term that was introduced by Skyrme in [2] to ensure the existence

of solitonic solutions. In a generally covariant form the model is defined by the action integral

$$\mathcal{S}[U] = \int \sqrt{-\det(g_{\mu\nu})} \mathrm{d}^4 x \left(\frac{1}{4} \mathbf{f}_\pi^2 \, Tr \left(C_\mu C^\mu \right) + \frac{1}{32\mathbf{e}^2} \, Tr \left([C_\mu, C_\nu][C^\mu, C^\nu] \right) \right),$$

where $C_\mu = iU^+\partial_\mu U$ is the Lie algebra valued topological four-current and $g_{\mu\nu}$ is the metric tensor.

Currently, apart from its successful applications to nuclear and condense matter physics, the Skyrme model, as an example of a nonlinear field-theoretical model, is utilized in studies of Einstein's field equations (for a review of similar topics you may consult [3]). The Einstein-Skyrme model [4] in the spherically symmetric case, thus more tractable and also physically most interesting, reduces to a dynamical system which possesses a very rich structure of solutions. The spectrum of soliton solutions consists of two branches which coalesce when coupling between the Skyrme's nonlinear matter and gravity is sufficiently large. For each winding number this happens at some critical value of the coupling constant which is a combination of the Newton's constant G and the dimensional constant $\mathbf{f}_\pi$ by which the nonlinear field couples to gravity. In the limit $G \to 0$ one obtains, among other solutions, flat space-time skyrmions. Due to complex structure of these equations it seems reasonable to investigate general properties of the system in a simpler setup. It turns out, that many characteristics of gravitating Skyrmions may be observed in the Skyrme model on the three sphere.

The key idea is to 'simulate' gravitational interactions by considering the Skyrme fields in a given space-time background. For static solutions we assume that the space-time is time invariant and has a constant and positive space-like curvature. Therefore, we assume that the three-sphere of radius R will play the role of our physical space. Its curvature may be thought as representing a gravitational field whose intensity is dependent on how strongly this space is curved. Therefore, we will see how curvature, and so 'gravity', affects spectra of solutions.

The Skyrme model on the three-sphere contains two natural scales of length, namely, the radius R of the base three-sphere and the characteristic soliton size $(\mathbf{f}_\pi \mathbf{e})^{-1}$, which break scale invariance. These scales constitute a free dimensionless parameter whose value is crucial to the number of possible solutions. These solutions are topological solitons, that is, localized, finite energy field configurations with nontrivial topology.

Due to the existence of the free parameter this model generates a denumerable and infinite spectrum of real numbers $1/\alpha$ which can be interpreted as critical radii (in units where $\mathbf{ef}_\pi = 1$) of the base three-sphere, below which some solutions cease to exist. In a sense, this phenomenon is analogous to the formation of black-hole solutions in the Einstein-Skyrme model. The base three-sphere may be too small to support some solutions since curvature may be too high (i.e. 'gravitational' field too strong). Indeed, playing with R, $\mathbf{e}$ and $\mathbf{f}_\pi$ we can construct the 'gravitational constant' $G(R) = 1/(R\mathbf{ef}_\pi^2)^2$ (which is proportional to the Gauss curvature of our three-space), then a soliton which has the characteristic mass $\mathbf{f}_\pi/\mathbf{e}$ can not exist when its size is comparable with its 'Schwarzschild radius' - the characteristic solitonic size $1/(\mathbf{ef}_\pi)$. In this special case it happens when $R\mathbf{ef}_\pi = \sqrt{2}$ which turns out to be of the same rank (actually this happens accidentally to be exactly the same value) of the radius of the three-sphere below which the 1-skyrmion cease to exist. We also mention the fact, first observed for gravitating solitons in [4], that the lowest eigenvalue of the Hessian of the counterpart of the flat space stable soliton decreases as α grows and tends to zero as α tends to some critical α_o at which the branch of gravitating skyrmions coalesces with another branch of unstable solitons. This catastrophe-theoretical phenomenon is also observed in our model and, as it will be shown, can be astonishingly simply tackled analytically.

2 Some analytical results

The Skyrme field may be equivalently considered as a map from the metrical base space, which is the space-time $(\mathcal{M}, g)$ (with metric g), to the target space $(\mathcal{N}, G)$ (with metric G) – here the $SU(2)$ group. The $SU(2)$ group is the metrical and topological unit three-sphere. For any map between metrical manifolds $(\mathcal{M}, g) \ni x \to y(x) \in (\mathcal{N}, G)$ one can construct the Jacobi matrix $J_\alpha^a = \partial_\alpha y^a$ which pulls back the metric G from $\mathcal{N}$ to $\mathcal{M}$ via the mapping $\hat{G}_{\alpha\beta} = J_\alpha^a J_\beta^b G_{ab}$. The tensor $\hat{G}$ can be coupled with g according to the definition $\hat{g}_{\mu\nu} = g_{\mu\nu} + \kappa^2 \hat{G}_{\mu\nu}$, thus $\hat{g}$ may serve as another metric tensor on $\mathcal{M}$. This provides us with several invariants of $\hat{G}$ which are obtained as the expansion coefficients of the function $\mathrm{Det}(A)(\kappa^2)$ with respect to κ^2, where $A_\beta^\alpha = g^{\alpha\gamma} \hat{g}_{\gamma\alpha}$. In the Skyrme model only the invariants $g^{\alpha\gamma}\hat{G}_{\gamma\alpha}$ (the so called sigma term) and $(g^{\alpha\gamma}\hat{G}_{\gamma\alpha})^2 - g^{\alpha\gamma}\hat{G}_{\gamma\delta}g^{\delta\mu}\hat{G}_{\mu\alpha}$ (the Skyrme term) are used to construct the Lagrangian. We impose on solutions the hedgehog ansatz, that is, for static and spherically symmetric solutions we require $\Psi = F(\psi)$, $\Theta = \theta$ and $\Phi = \phi$, where (Ψ, Θ, Φ) are spherical angles on the unit target three-sphere (the manifold of the $SU(2)$ group), while (ψ, η, ϕ) are angles on the base three-sphere of radius L. Hence, such solutions are critical points of the energy functional

$$\mathcal{U}[F] = \int_0^\pi 4\pi \sin^2\psi \, d\psi \left\{ L\left[F'(\psi)^2 + 2\frac{\sin^2 F(\psi)}{\sin^2\psi} \right] + \frac{1}{L}\left[2F'(\psi)^2 + \frac{\sin^2 F(\psi)}{\sin^2\psi} \right] \frac{\sin^2 F(\psi)}{\sin^2\psi} \right\}, \quad (1)$$

where L is the radius of the base three-sphere, and the unit of energy and length are respectively $\mathbf{f}_\pi \mathbf{e}^{-1}/2$ and $(\mathbf{e}\mathbf{f}_\pi)^{-1}$. The ansatz has symmetries of the full Lagrangian (that is, without any symmetry constraints) thus, according to the principle of symmetric criticality[5], minima of the reduced functional (1) are also critical points of the full action. Therefore, our problem reduces to finding solutions of the equation

$$\left(L + \frac{2}{L}\frac{\sin^2 F(\psi)}{\sin^2\psi} \right) \sin^2\psi F''(\psi) + \left(L + \frac{1}{L}\frac{\sin 2F(\psi)}{\sin 2\psi}F'(\psi) \right) \sin 2\psi F'(\psi)$$
$$- \left(L + \frac{1}{L}\frac{\sin^2 F(\psi)}{\sin^2\psi} \right) \sin 2F(\psi) = 0. \quad (2)$$

Solutions of this equation were originally analyzed in[6,7] (where also a nice geometrical point of view on the Skyrme model was presented). Full spectrum of these solutions was studied in[8] and their stability analysis was carried out in[9]. A special attention we pay to energetical stability analysis of the only solution known analytically, that is, the identity solution $F(\psi) = \psi$ which we denote by $\mathbf{H}_1$. This is the only solution which exist for $L < \sqrt{2}$ in the topological sector $Q = 1$ (Q is an integer, the topological charge, and is defined here for finite energy solutions by $F(\pi) - F(0) = Q\pi$). For $L > \sqrt{2}$ and sufficiently small, another solution exist, whose energy is finite and tends, as $L \to \infty$, to the energy of the flat-space 1-skyrmion. We denote the solution by $\mathbf{S}_1$. This solution bifurcates from $\mathbf{H}_1$ at the critical radius $L = \sqrt{2}$ and appears due to instability of $\mathbf{H}_1$. To understand this fact, it suffices to analyze the behaviour of the energy functional $\mathcal{U}[F + \epsilon\xi]$ at $F(\psi) = \psi$, where ϵ is a small number and $\xi(\psi)$ are spherically symmetric perturbations which vanish on boundaries, that is, for which $\xi(0) = \xi(\pi) = 0$. Since $F(\psi) = \psi$ is a solution, the first variation vanishes $\delta\mathcal{U}[F](\xi) = 0$. Therefore, we examine the second variation which reads

$$\delta^2\mathcal{U}[F](\xi, \xi) = \int_0^\pi 4\pi \sin^2\psi \left\{ \left(L + \frac{2}{L}\frac{\sin^2 F}{\sin^2\psi} \right) \xi'^2 + \frac{4}{L}\frac{\sin 2F}{\sin^2\psi}F'\xi\xi' \right\}$$
$$+ \int_0^\pi 4\pi \sin^2\psi \left\{ \left[\frac{2}{L}(1 + 2\cos 2F)\frac{\sin^2 F}{\sin^4\psi} + \frac{2\cos 2F}{\sin^2\psi}\left(L + \frac{F'^2}{L} \right) \right] \xi^2 \right\}. \quad (3)$$

The general theorem due to Hilbert [10] states that the consecutive minima of the functional above, at $F(\psi) = \psi$, are solutions of the equation

$$-(\sin^2 \psi \xi'(\psi))' + 2\left(1 - 2\frac{L^2 + 1}{L^2 + 2}\sin^2 \psi\right)\xi(\psi) = \frac{L}{2 + L^2}\sin^2 \psi \lambda \xi(\psi)$$

which is the Euler-Lagrange equation $\delta_\xi(\delta^2 \mathcal{U}[F](\xi, \xi)) = 0$. These solutions span a complete and denumerable linear space of functions ξ_n which are mutually orthogonal with respect to the scalar product $g(u, v) = 4\pi \int_0^\pi u(\psi)v(\psi)\sin^2 \psi \mathrm{d}\psi$. The λ_n are the corresponding eigenvalues given by

$$\lambda_n = \frac{2}{L}(n^2 + 4n + 1) + L(n^2 + 4n - 1), \quad n = 0, 1, 2, \ldots$$

and they are positive for all L with the exception of λ_0, which is positive for $L < \sqrt{2}$ and negative for $L > \sqrt{2}$. The corresponding (unnormalized) eigenfunctions read $\xi_{2n}(\psi) = \sum_{k=0}^{n}(2k + 1)\sin((2k + 1)\psi)$ and $\xi_{2n+1}(\psi) = \sum_{k=1}^{n+1} 2k\sin(2k\psi)$. Thus, for $L > \sqrt{2}$, $\mathbf{H}_1$ is no longer the absolute minimum of the energy functional (1) in the sector $Q = 1$, since then the perturbation $\xi_0 = \sin\psi$ decreases energy of $\mathbf{H}_1$. Due to the fact that ξ_n are linear combinations of sine functions of the argument $k\psi$, the alleged global minimum $\mathbf{S}_1$ can be decomposed into a series of these functions. By substituting into (2) one obtains, by formal Taylor expansion about $L = \sqrt{2}$, that the profile of $\mathbf{S}_1$ on the right of $L = \sqrt{2}$ and L sufficiently small, is given by

$$F(\psi, x) = \psi + x\sin\psi + \frac{3}{20}x^2 \sin 2\psi - x^3\left(\frac{29369}{316800}\sin\psi - \frac{11}{480}\sin 3\psi\right) + o(x^3),$$

and the corresponding energy reads

$$\frac{\mathcal{U}[F]}{12\pi^2}(x) = \frac{3\sqrt{2}}{4}\left(1 + \frac{11}{180}x^2 - \frac{209}{10800}x^4 + \frac{5209}{864000}x^6\right) + O(x^8), \quad x^2 = \sqrt{\frac{60}{11}\left(\frac{L}{\sqrt{2}} - 1\right)},$$

which is less then the energy of $\mathbf{H}_1$. The series, if divergent for large x, should have its analytical continuation and reproduce the energy of the flat space 1-skyrmion known from the original Skyrme model. As $x \searrow 0$ the profile of $\mathbf{S}_1$ smoothly coalesces with that of $\mathbf{H}_1$. Note that $\lim_{x \searrow 0}\partial_x F(\psi, x) = \sin\psi$ which is the mode of instability of $\mathbf{H}_1$. It is worth to note also, that such expansion procedure fails on the left of the critical radius $L = \sqrt{2}$.

This shows the mechanism how solutions may appear at some critical parameters of a coupling constant of a theory and not exist if the coupling is too large. In the case of the 1-skyrmion this phenomenon could be analyzed exactly.

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